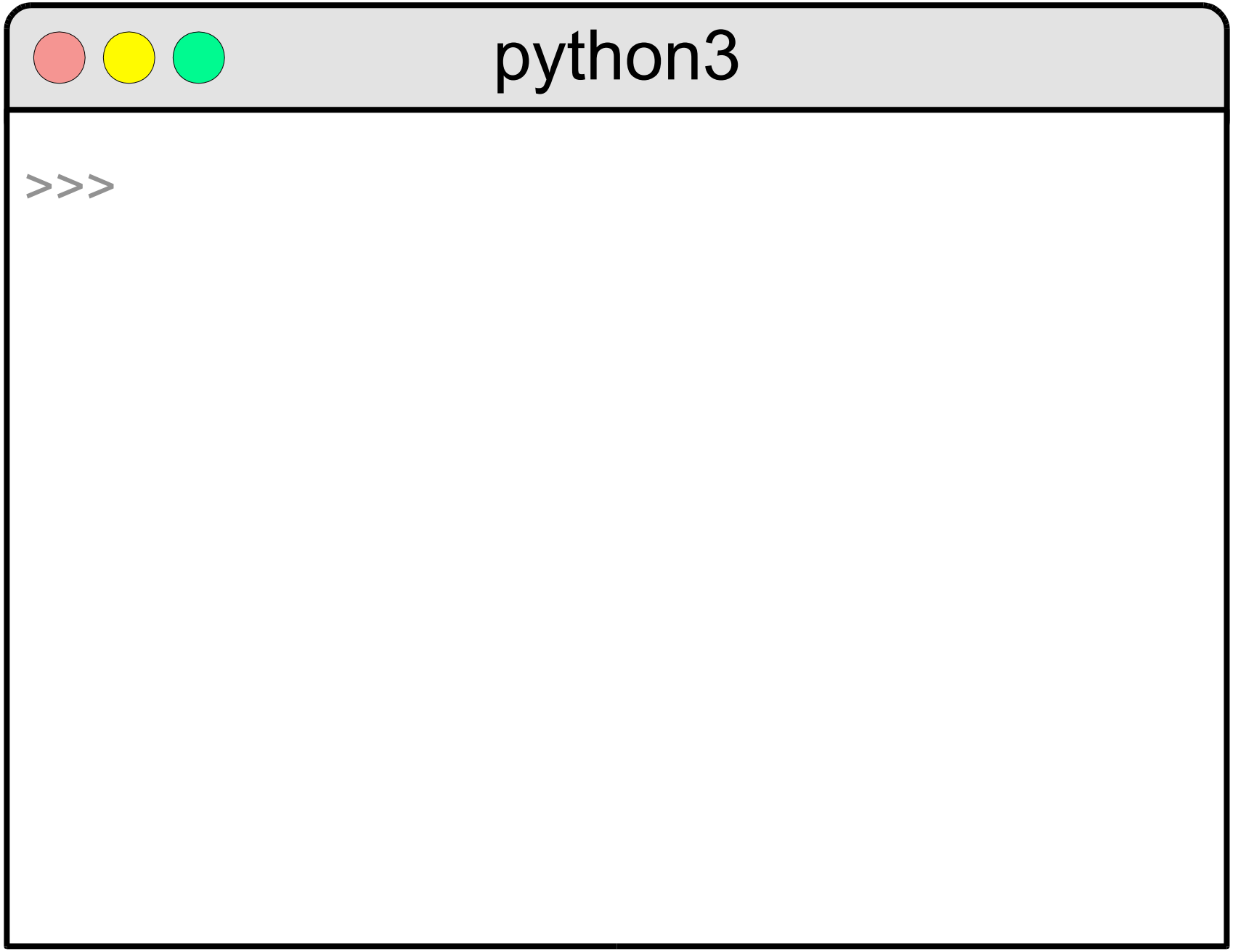
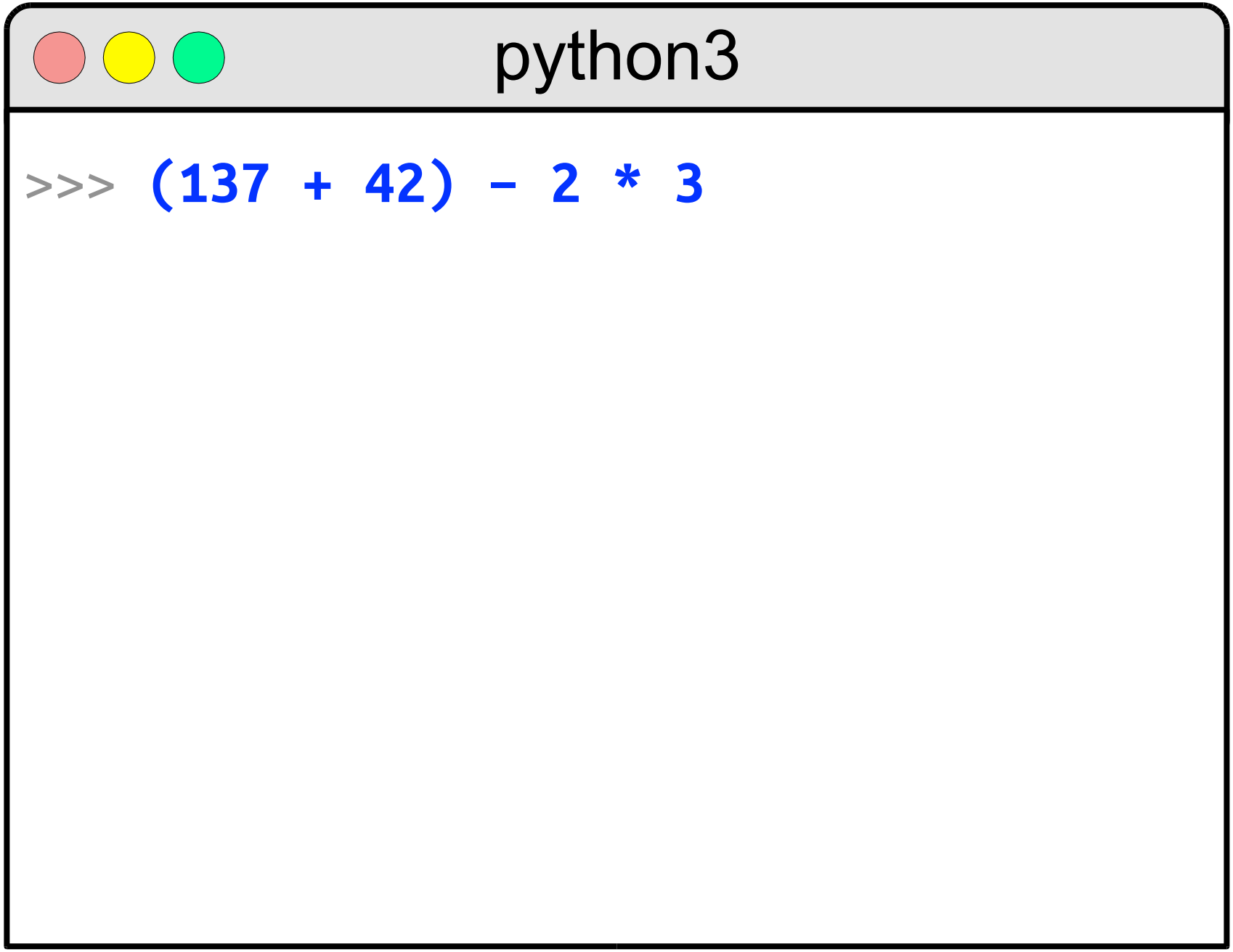
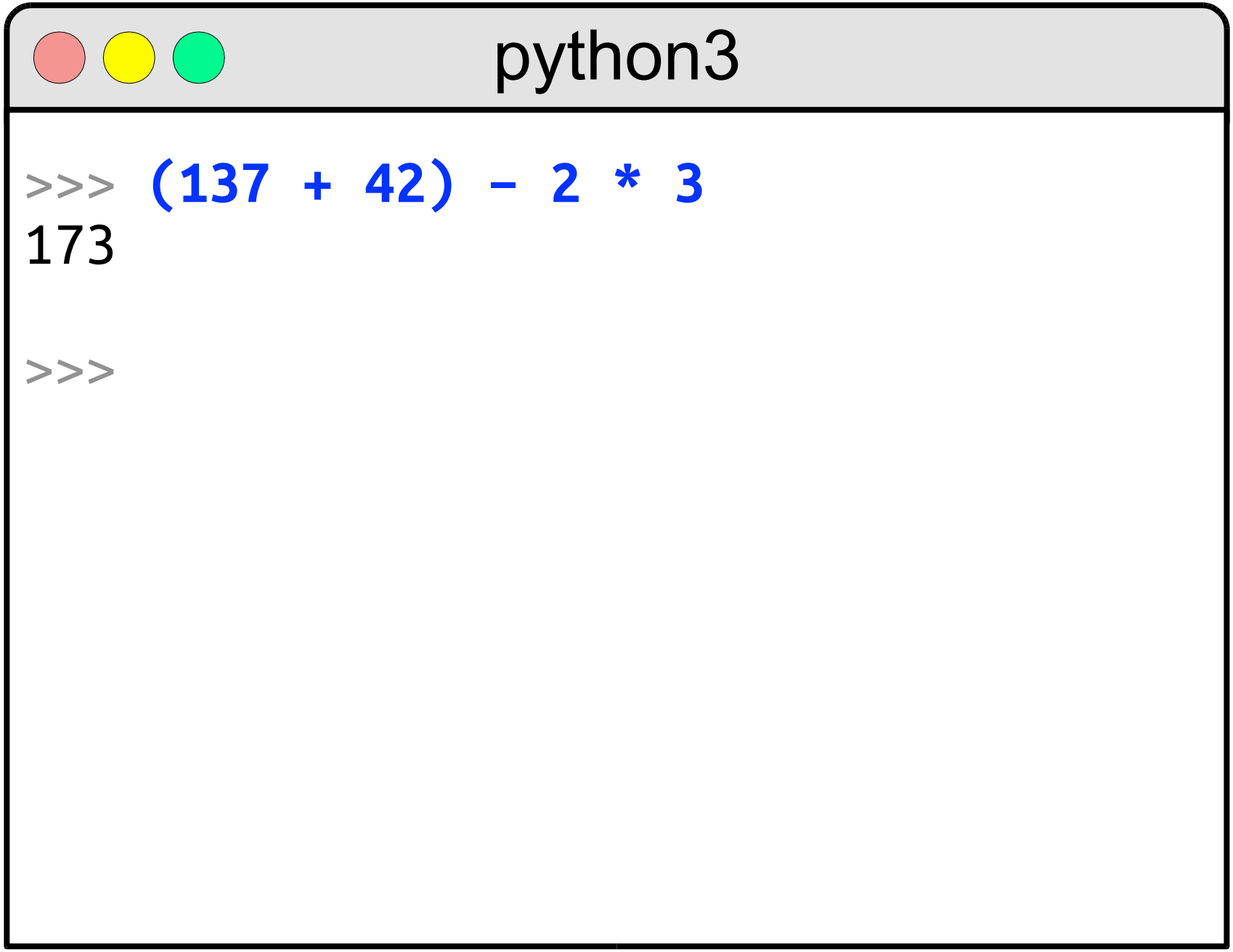
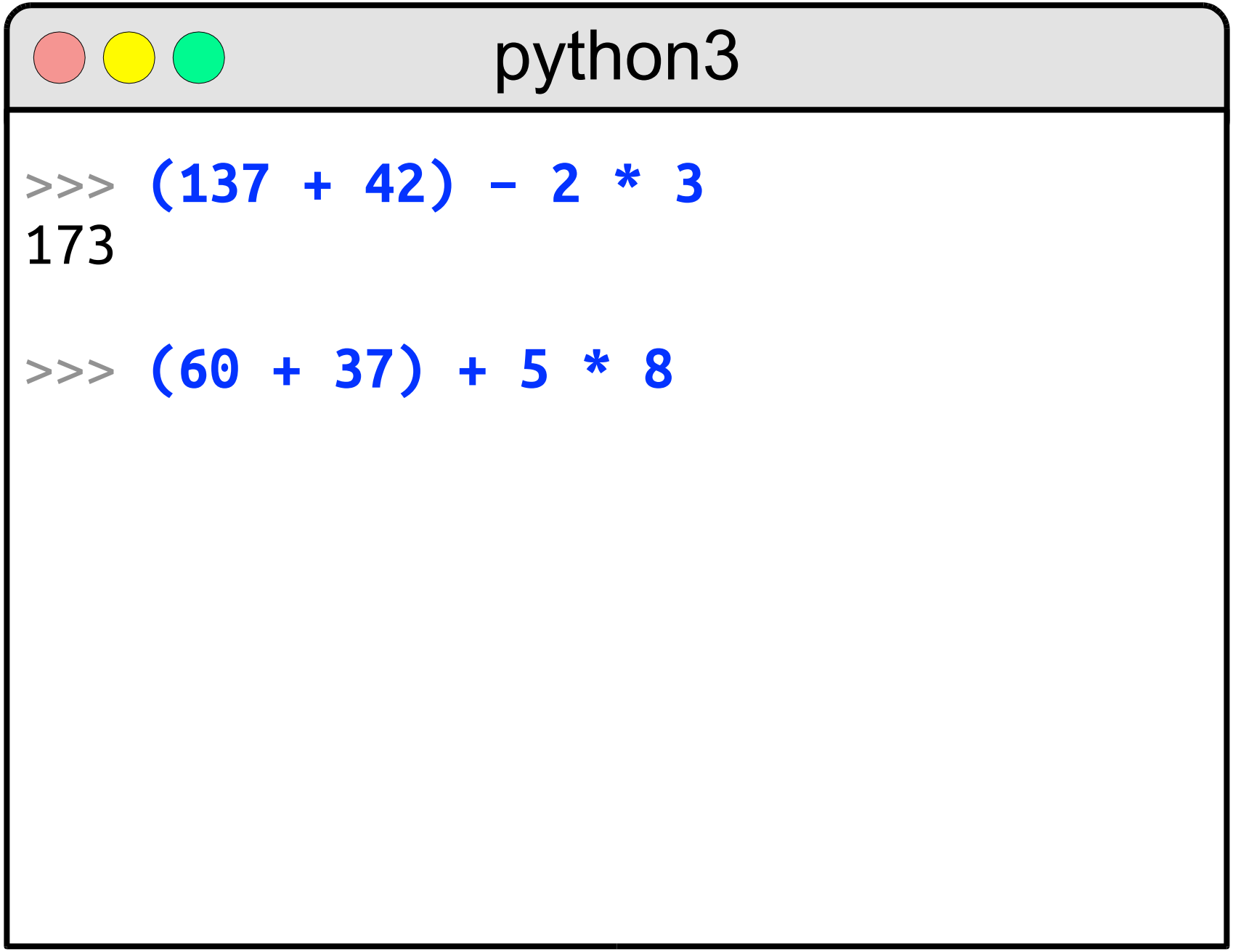
Context-Free Grammars

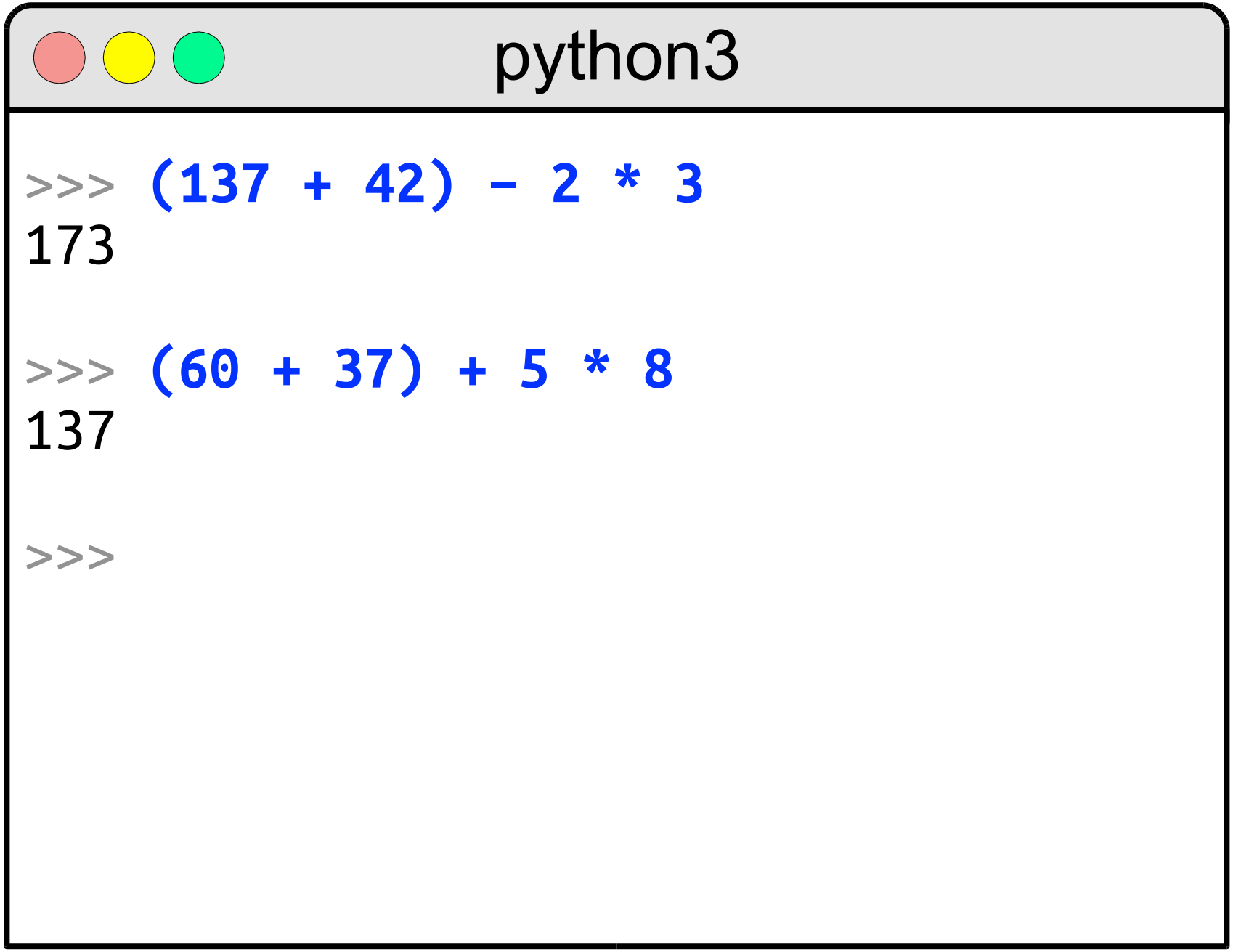
A Motivating Question


python3
>>>


python3
>>> (137 + 42) - 2 * 3


python3
>>> (137 + 42) - 2 * 3
173
>>>


python3
>>> (137 + 42) - 2 * 3
173
>>> (60 + 37) + 5 * 8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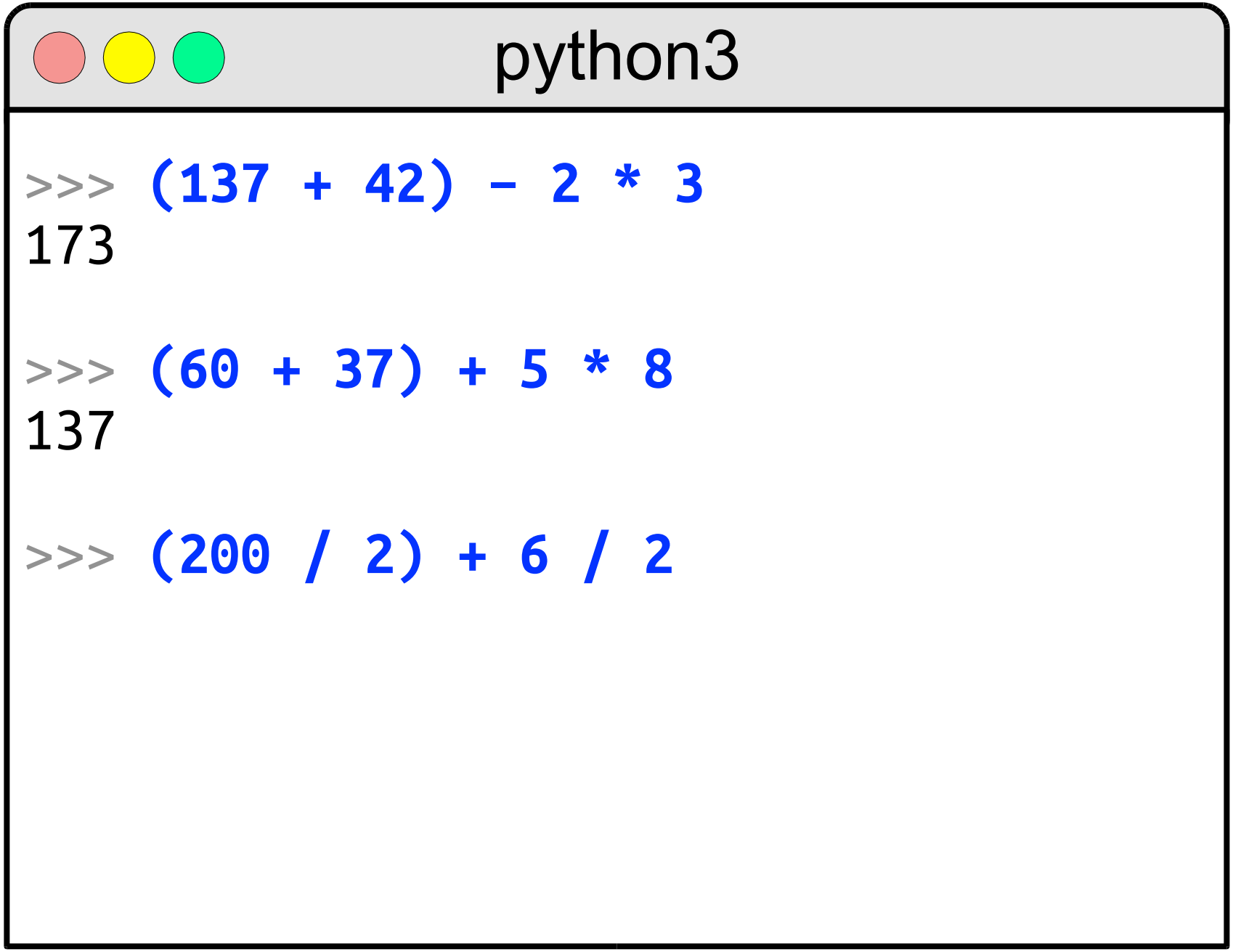
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>>> (137 + 42) - 2 * 3
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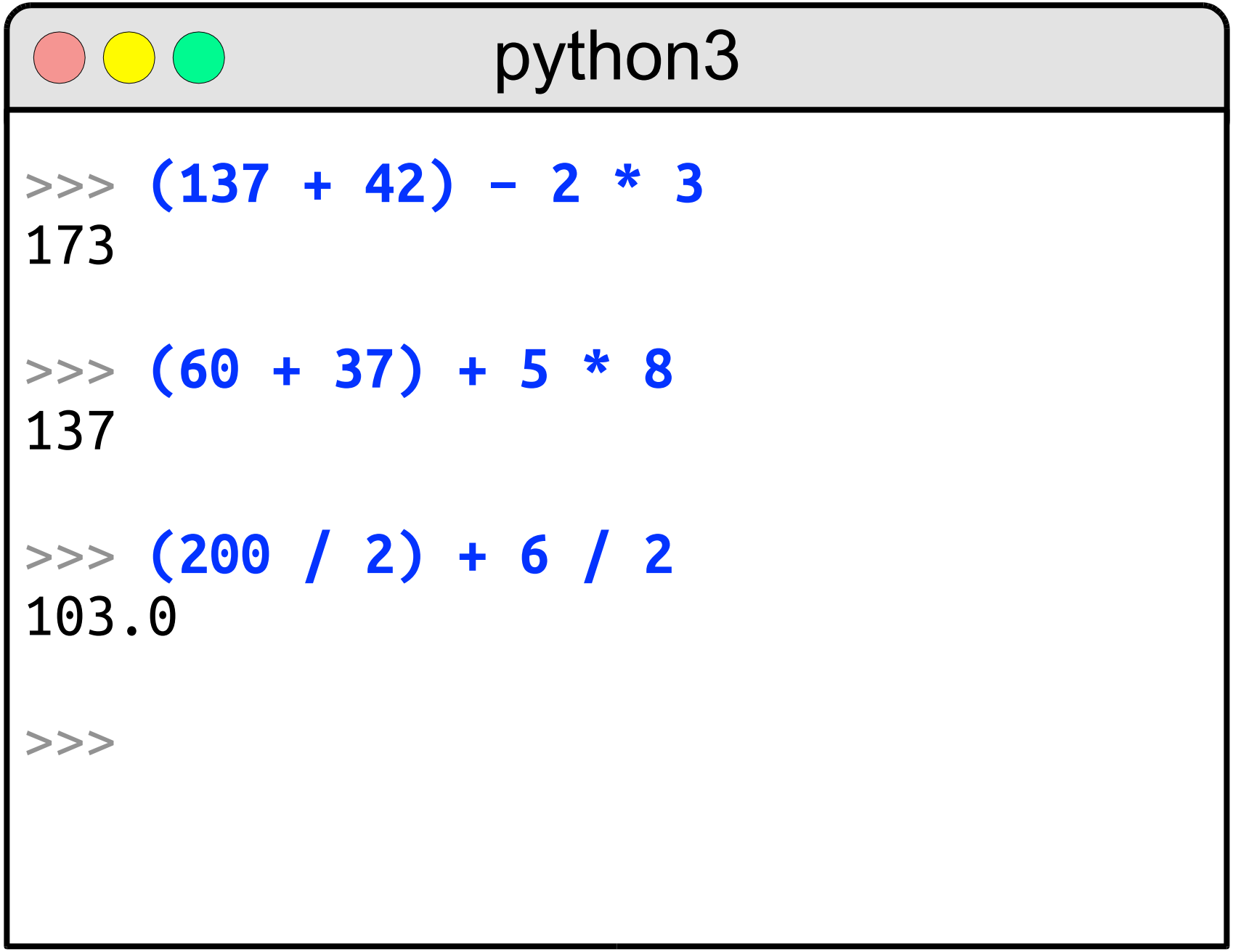
python3
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173
>>> (60 + 37) + 5 * 8
137
>>> (200 / 2) + 6 / 2
103.0
>>>

Mad Libs for Arithmetic

(____ ____ ____) ____ ____ ____ ____

Int Op Int Op Int Op Int

Slide credit: Amy Liu

Mad Libs for Arithmetic

(	26	+	42	)	*	2	+	1
	Int	Op	Int		Op	Int	Op	Int

Slide credit: Amy Liu

Mad Libs for Arithmetic

(____ ____ ____) ____ ____ ____ ____

Int Op Int Op Int Op Int

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Mad Libs for Arithmetic

(	7	*	5	)	/	5	-	49
	Int	Op	Int		Op	Int	Op	Int

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Mad Libs for Arithmetic

(_____ _____ _____) _____ _____ _____ _____

Int Op Int Op Int Op Int Op Int

This only lets us make arithmetic expressions of the form **(Int Op Int) Op Int Op Int**.

What about arithmetic expressions that don't follow this pattern?

Slide credit: Amy Liu

Recursive Mad Libs

Expr

Recursive Mad Libs

Expr

What can an arithmetic expression be?

Recursive Mad Libs

`int`

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What can an arithmetic expression be?

`int`	A single number.

Recursive Mad Libs

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Expr Op Expr	Two expressions joined by an operator.

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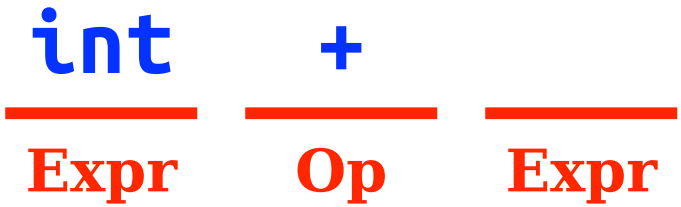

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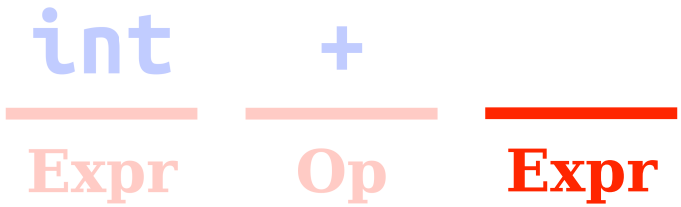

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Recursive Mad Libs

`int`	`+`			
Expr	**Op**	**Expr**	**Op**	**Expr**

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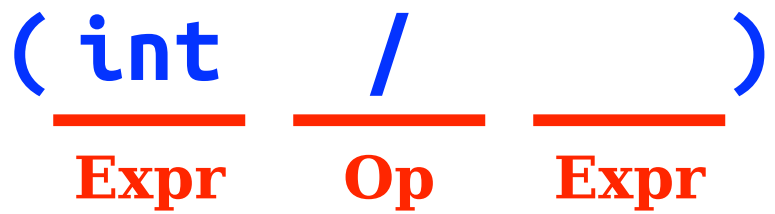

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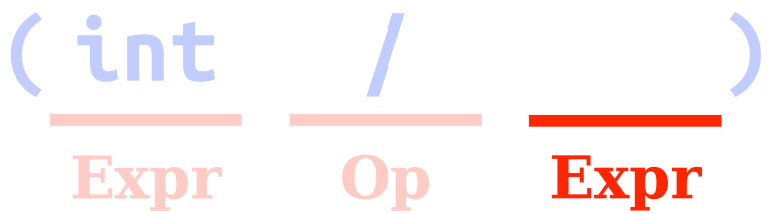

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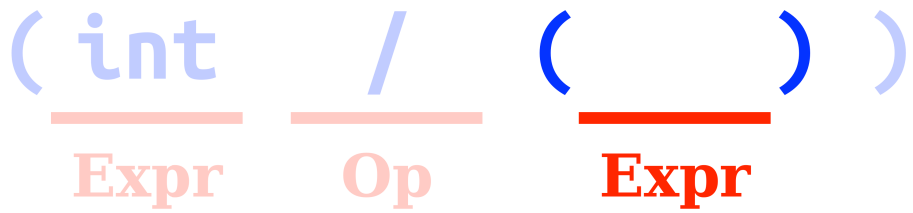

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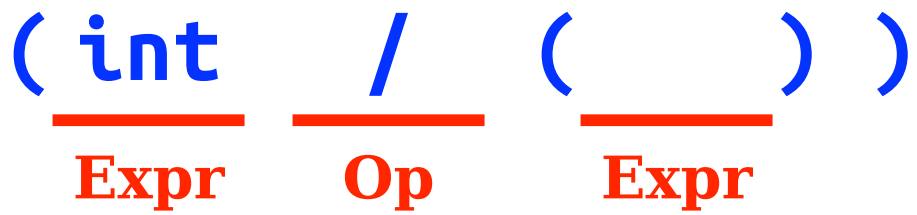

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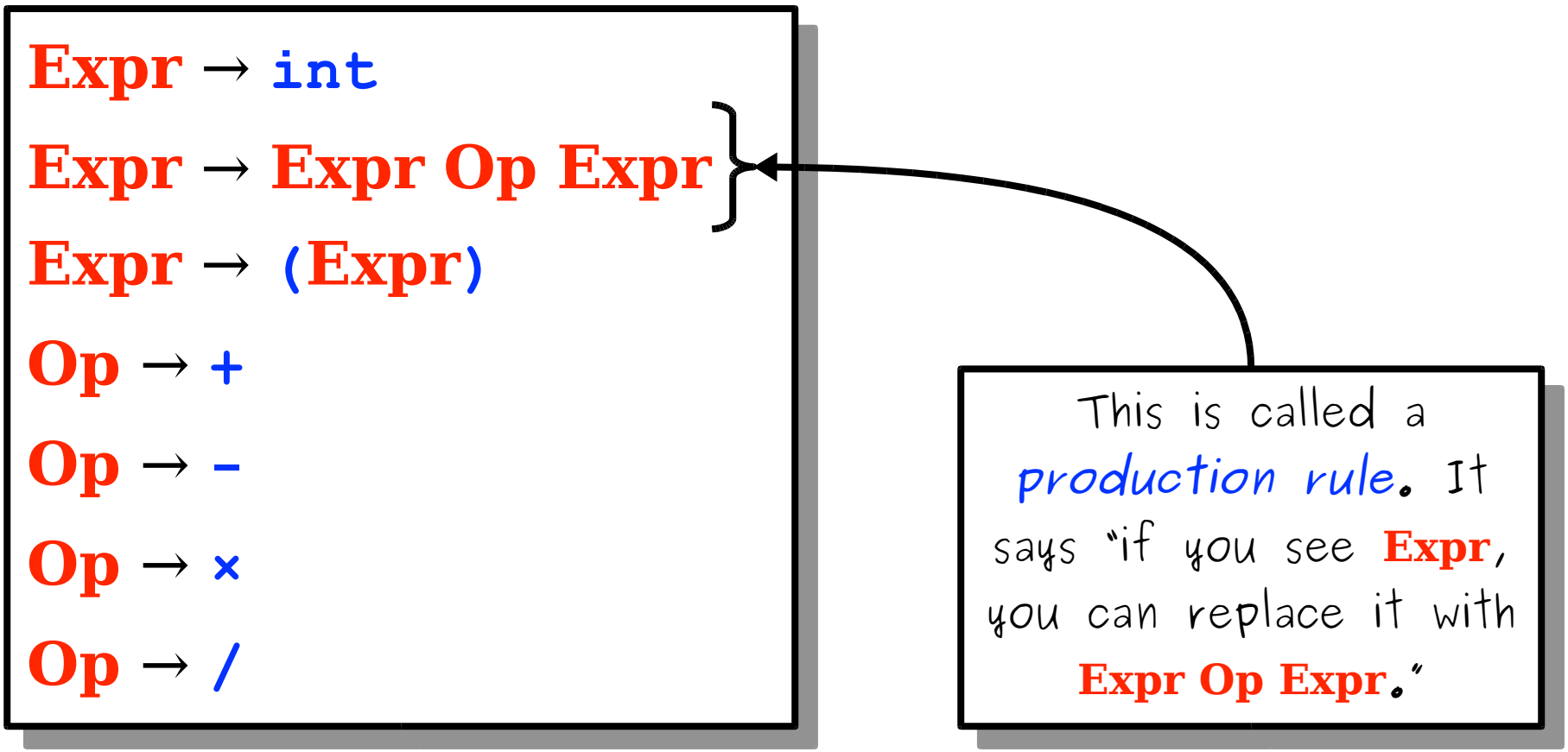
A ***context-free grammar*** (or ***CFG***) is a recursive set of rules that define a language.

(There's a bunch of specific requirements about what those rules can be; more on that in a bit.)

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.



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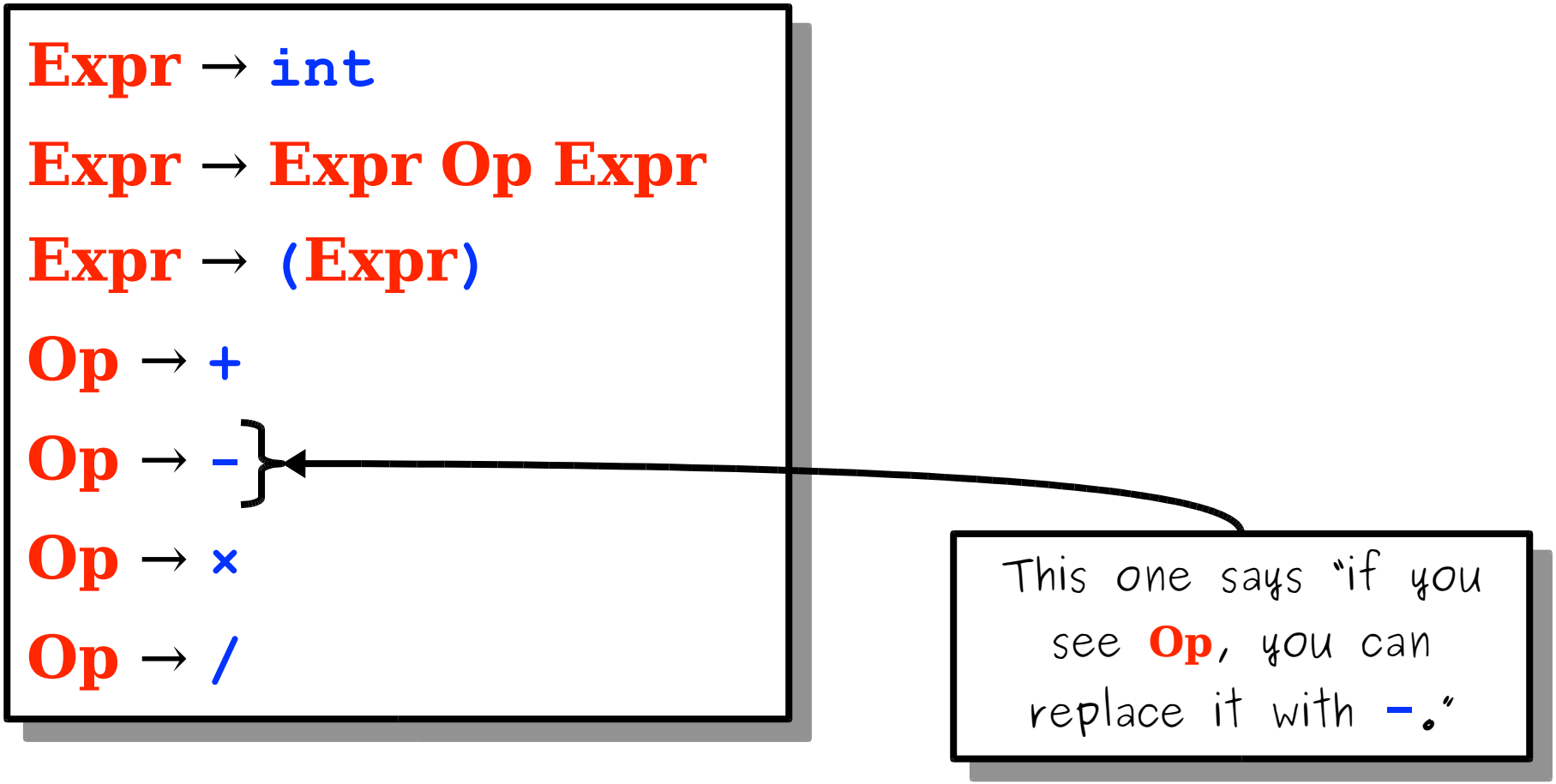

Arithmetic Expressions

- Here's how we might express the recursive rules from earlier as a CFG.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Expr
⇒ **Expr Op Expr**
⇒ **Expr Op** `int`
⇒ `int` **Op** `int`
⇒ `int / int`

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Expr
⇒ **Expr Op Expr**
⇒ **Expr Op** `int`
⇒ `int` **Op** `int`
⇒ `int / int`

These red symbols are called ***nonterminals***. They're placeholders that get expanded later on.

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Expr
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⇒ **Expr Op** `int`
⇒ `int` **Op** `int`
⇒ `int / int`

The symbols in blue monospace are **terminals**. They're the final characters used in the string and never get replaced.

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⇒ **Expr Op** `(`**Expr Op Expr**`)`
⇒ **Expr** `×` `(`**Expr Op Expr**`)`
⇒ `int` `×` `(`**Expr Op Expr**`)`
⇒ `int` `×` `(int` **Op Expr**`)`
⇒ `int` `×` `(int` **Op** `int)`
⇒ `int` `×` `(int + int)`

Context-Free Grammars

- Formally, a context-free grammar is a collection of four items:
  - a set of ***nonterminal symbols*** (also called ***variables***),
  - a set of ***terminal symbols*** (the ***alphabet*** of the CFG),
  - a set of ***production rules*** saying how each nonterminal can be replaced by a string of terminals and nonterminals, and
  - a ***start symbol*** (which must be a nonterminal) that begins the derivation. By convention, the start symbol is the one on the left-hand side of the first production.

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

Some CFG Notation

- In today's slides, capital letters in **Bold Red Uppercase** will represent nonterminals.
  - e.g. **A**, **B**, **C**, **D**
- Lowercase letters in `blue monospace` will represent terminals.
  - e.g. `t`, `u`, `v`, `w`
- Lowercase Greek letters in ***gray italics*** will represent arbitrary strings of terminals and nonterminals.
  - e.g. α, γ, ω
- You don't need to use these conventions on your own; just make sure whatever you do is readable. 😀

A Notational Shorthand

Expr → `int`

Expr → **Expr Op Expr**

Expr → `(`**Expr**`)`

Op → `+`

Op → `-`

Op → `×`

Op → `/`

A Notational Shorthand

Expr → `int` | **Expr Op Expr** | `(`**Expr**`)`

Op → `+` | `-` | `×` | `/`

Derivations

Expr

⇒ **Expr Op Expr**

⇒ **Expr Op (Expr)**

⇒ **Expr Op (Expr Op Expr)**

⇒ **Expr** `×` **(Expr Op Expr)**

⇒ `int × (`**Expr Op Expr**`)`

⇒ `int × (int` **Op Expr**`)`

⇒ `int × (int` **Op** `int)`

⇒ `int × (int + int)`

- A sequence of zero or more steps where nonterminals are replaced by the right-hand side of a production is called a ***derivation***.
- If string α derives string ω, we write $\alpha \Rightarrow^* \omega$.
- In the example on the left, we see that

  Expr $\Rightarrow^*$ `int × (int + int)`.

Expr → `int` | **Expr Op Expr** | **(Expr)**

Op → `+` | `-` | `×` | `/`

The Language of a Grammar

- If G is a CFG with alphabet Σ and start symbol **S**, then the ***language of G*** is the set

$$\mathcal{L}(G) = \{ \omega \in \Sigma^* \mid \mathbf{S} \Rightarrow^* \omega \}$$

- That is, $\mathcal{L}(G)$ is the set of strings of terminals derivable from the start symbol.

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Consider the following CFG G over $\Sigma = \{$**a**, **b**, **c**, **d**$\}$:

Q → **Qa** | **dH**
H → **bHb** | **c**

Which of the following strings are in $\mathcal{L}(G)$?

dca
dc
cad
bcb
dHaa

Answer at ***https://cs103.stanford.edu/pollev***

Context-Free Languages

- A language L is called a ***context-free language*** (or CFL) if there is a CFG G such that $L = \mathcal{L}(G)$.
- Questions:
  - How are context-free and regular languages related?
  - How do we design context-free grammars for context-free languages?

Context-Free Languages

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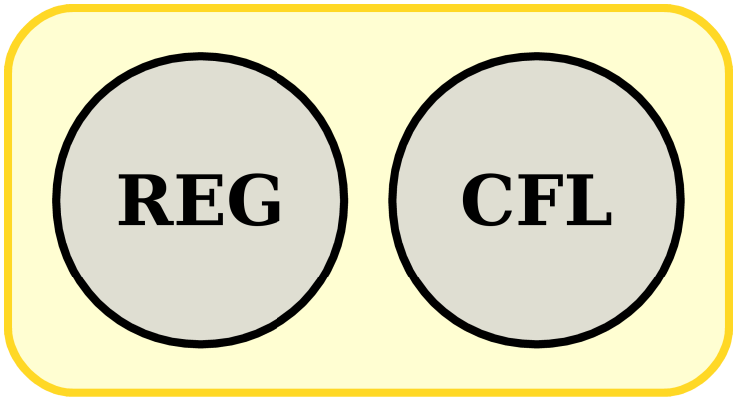
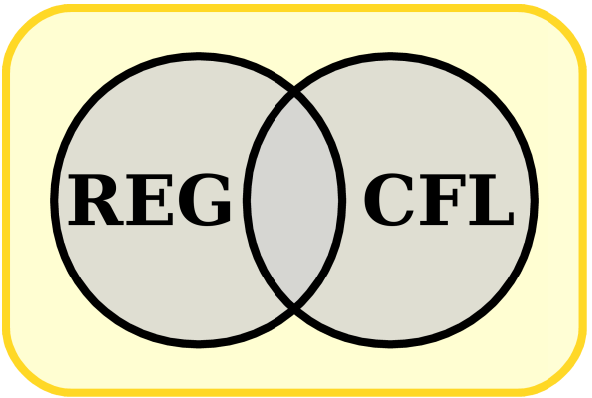
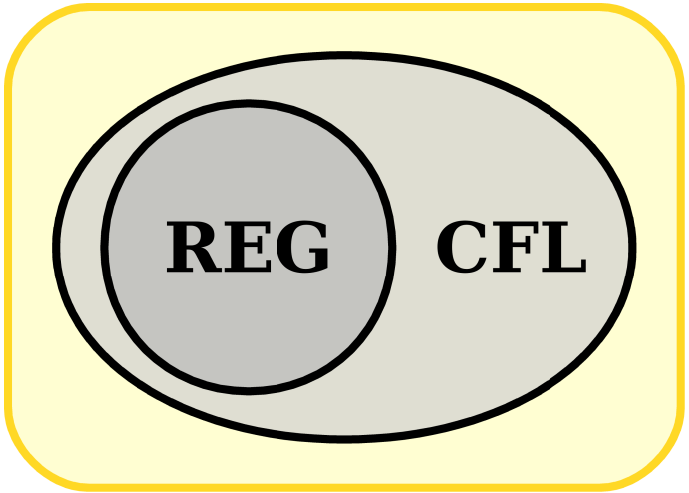
- How are context-free and regular languages related?

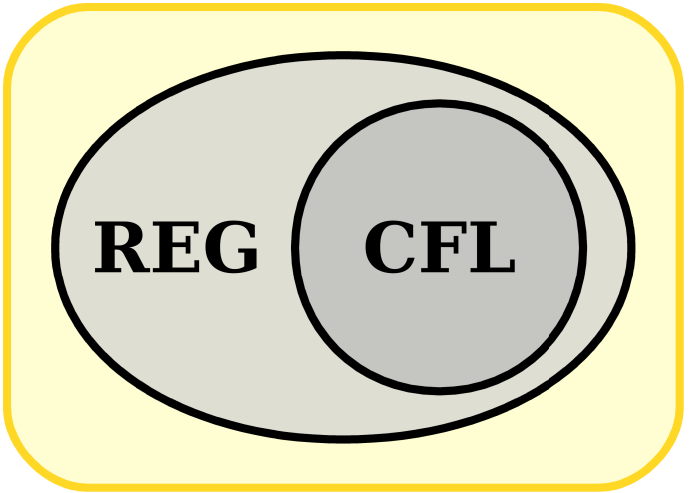
  How do we design context-free grammars for context-free languages?

Five Possibilities









CFGs and Regular Expressions

- CFGs consist purely of production rules of the form **A** → ω. They do not have the regular expression operators * or ∪.
- You can use the symbols * and ∪ if you'd like in a CFG, but they just stand for themselves.
- Consider this CFG G:

S → `a*b`

- Here, $\mathcal{L}(G) = \{$`a*b`$\}$ and has cardinality one. That is, $\mathcal{L}(G) \neq \{$ `a`^{n}`b` $\mid n \in \mathbb{N}\}$.

CFGs and Regular Expressions

- ***Theorem:*** Every regular language is context-free.
- ***Proof idea:*** Show how to convert an arbitrary regular expression into a context-free grammar.

a (b ∪ ε) c

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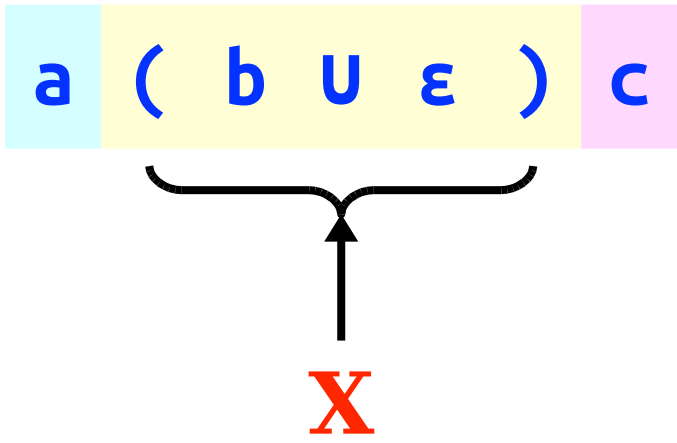

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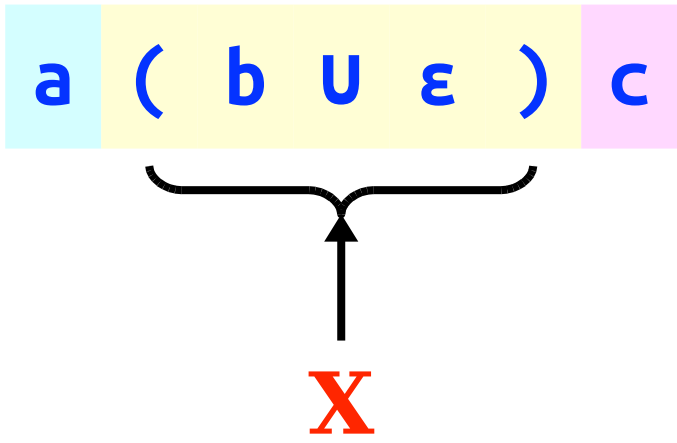

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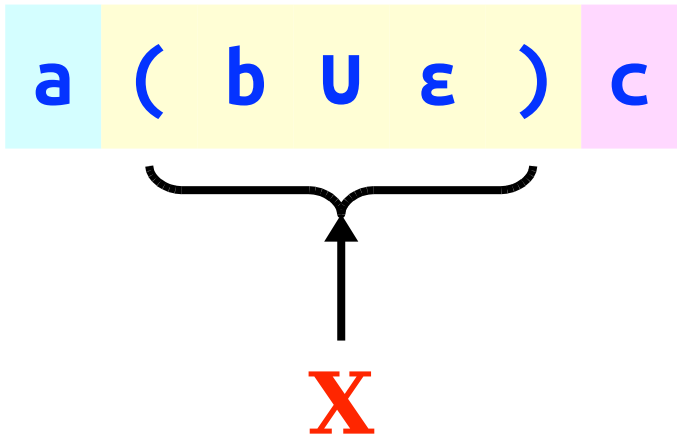

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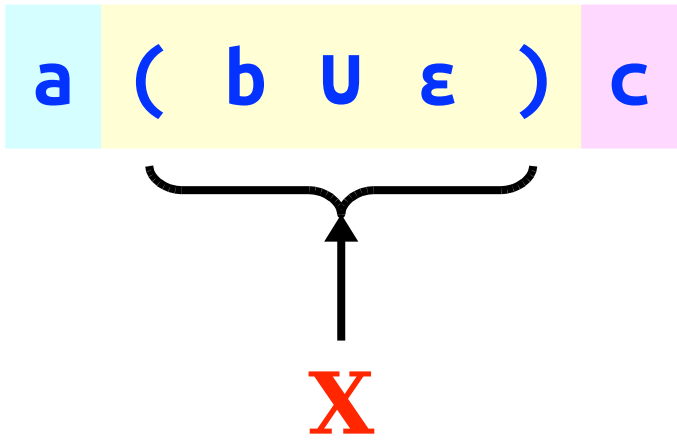
It's totally fine for a production to replace a nonterminal with the empty string.



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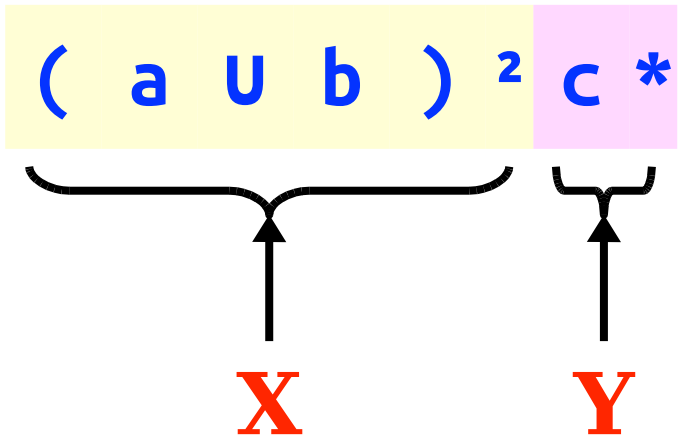

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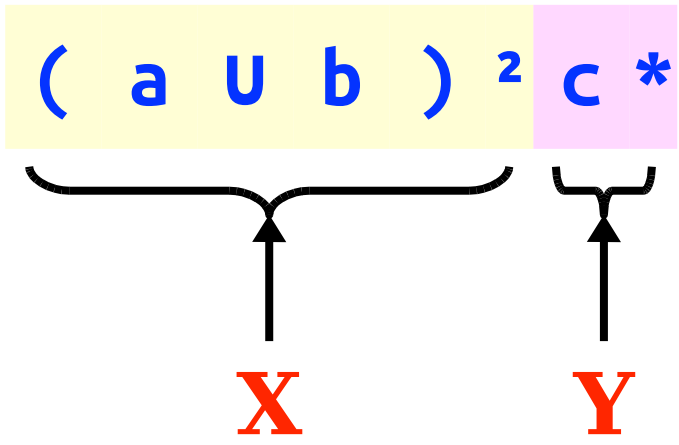
S → XY



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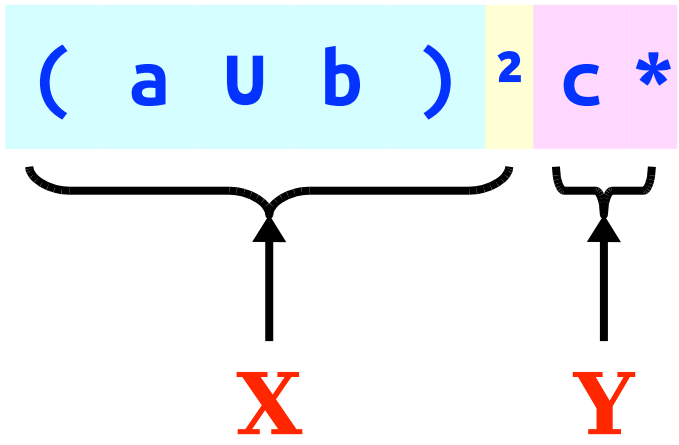

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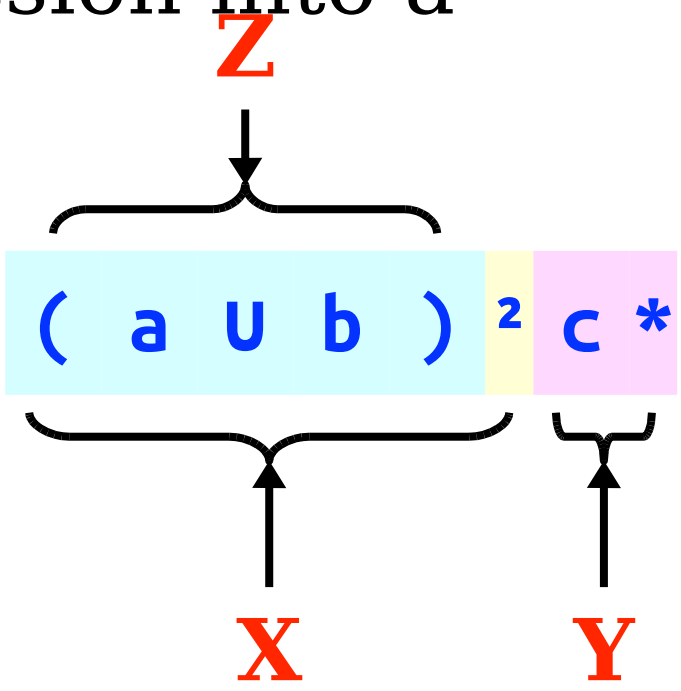

CFGs and Regular Expressions

- ***Theorem:*** Every regular language is context-free.
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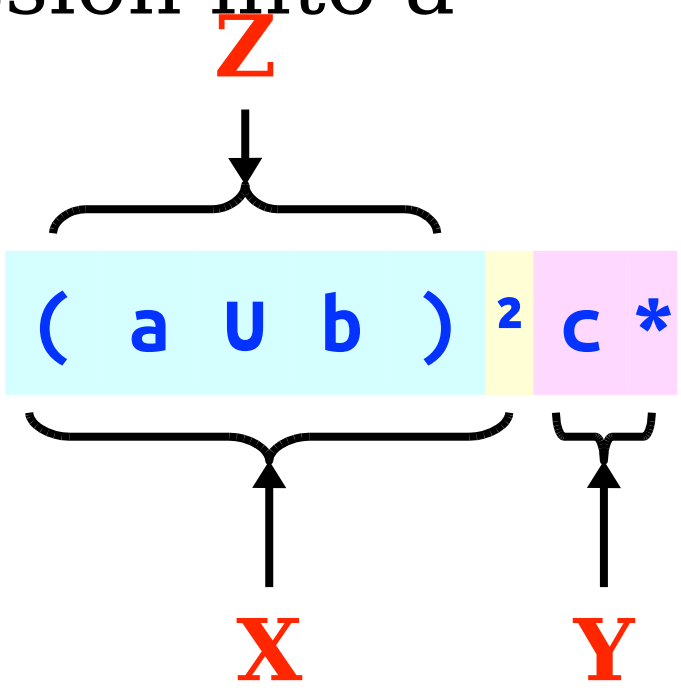

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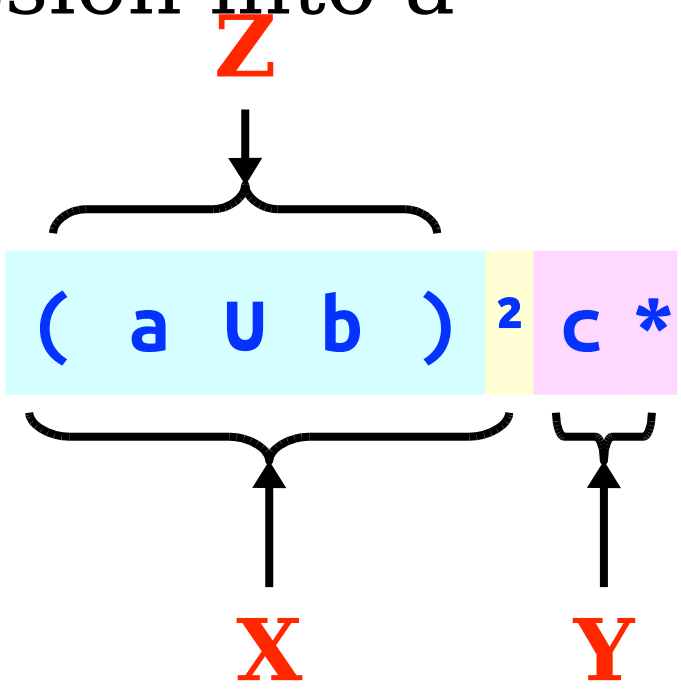

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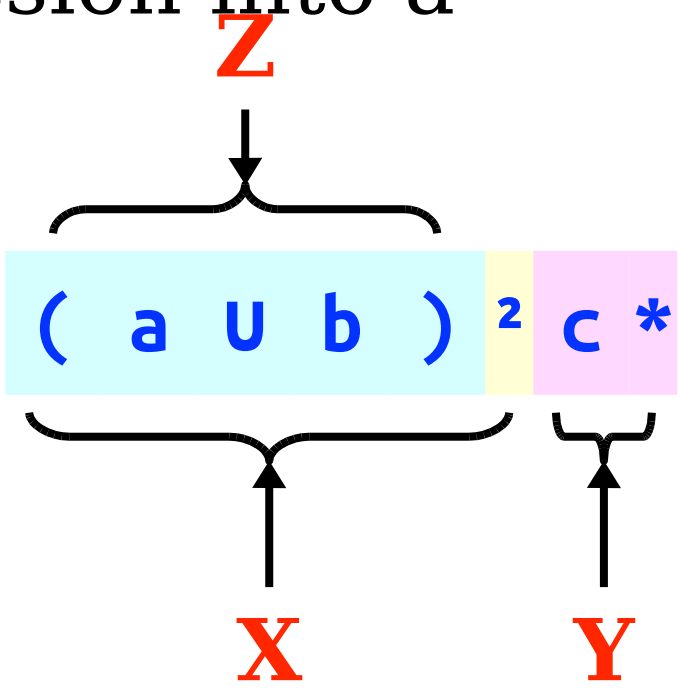
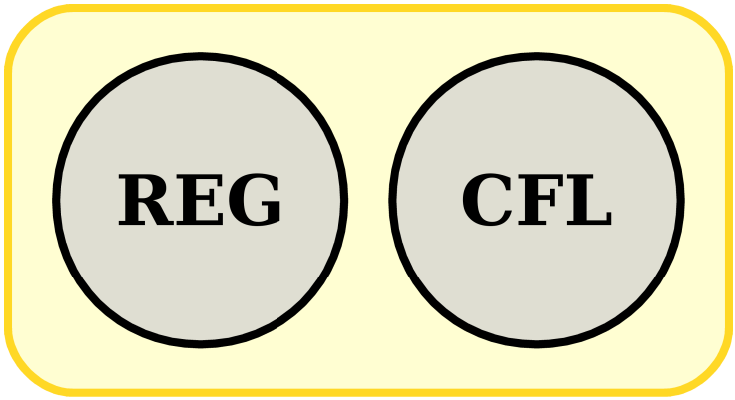
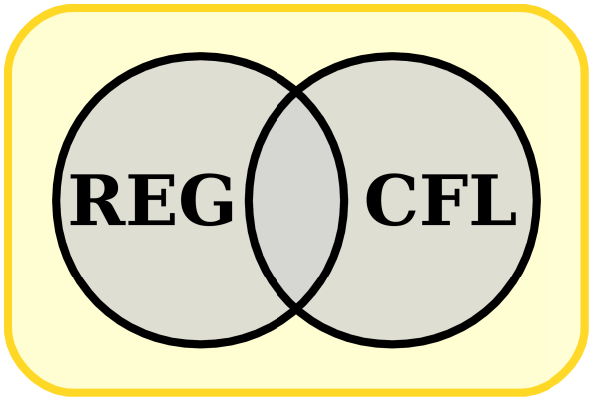
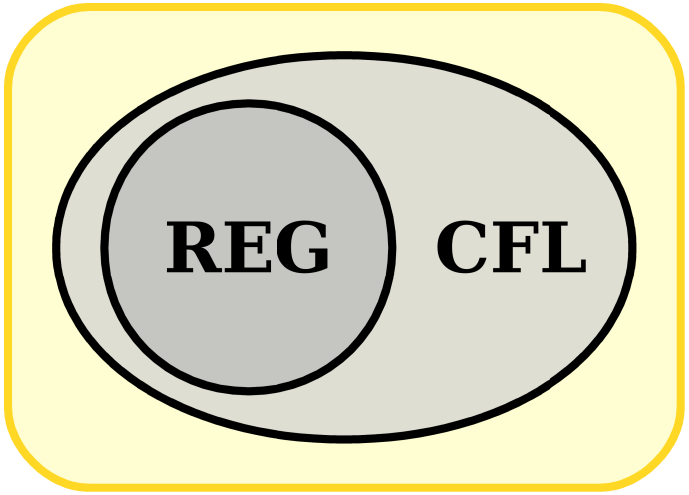
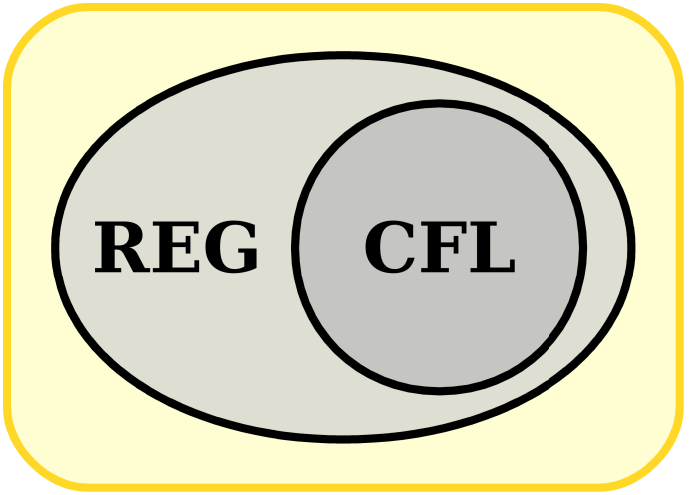


Two ~~Five~~ Possibilities









The Language of a Grammar

- Consider the following CFG G:

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b} \mid \varepsilon$$

- What strings can this generate?

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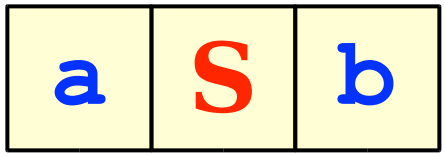

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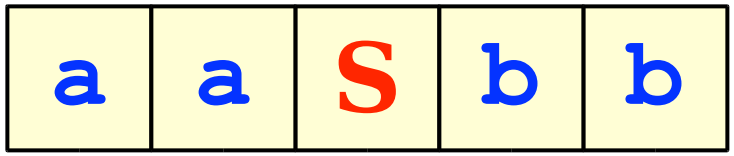

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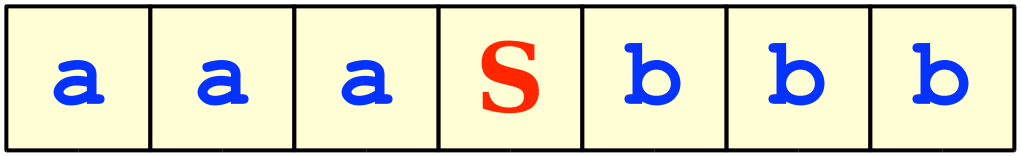

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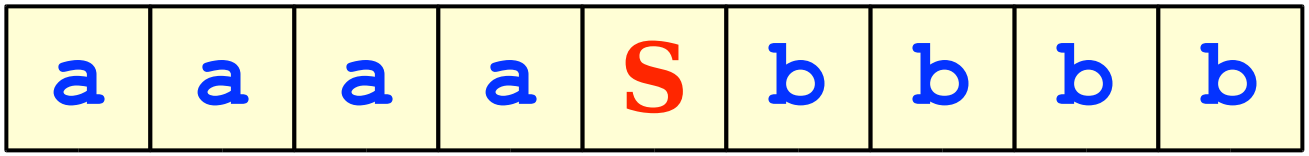

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b	b	b	b

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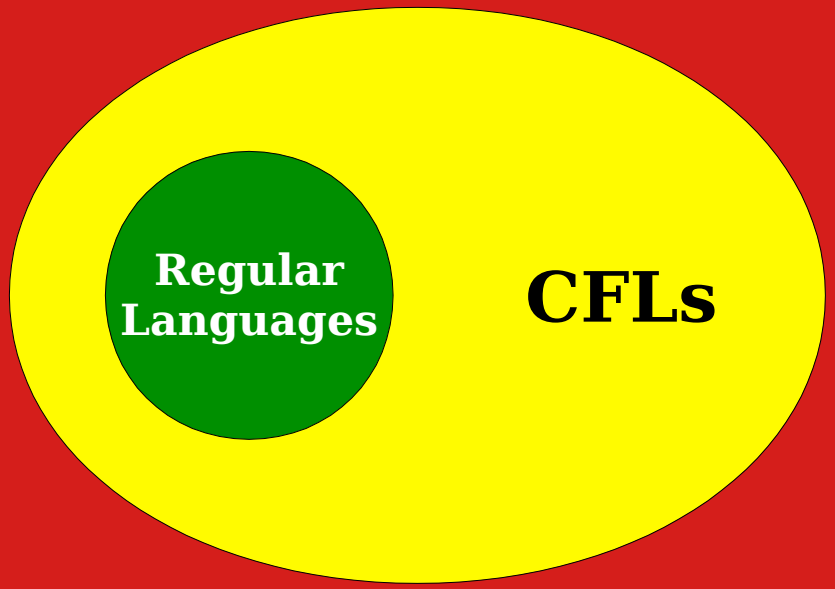
$$\mathcal{L}(G) = \{\ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\ \}$$



All Languages

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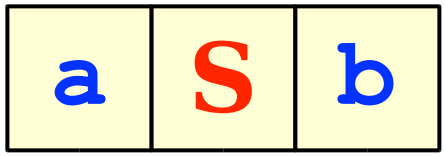

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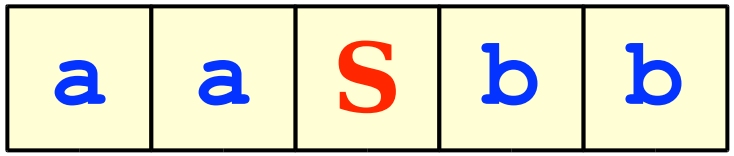

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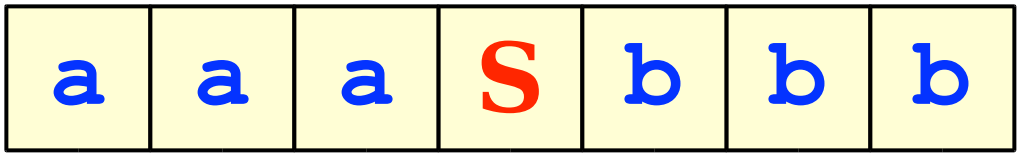

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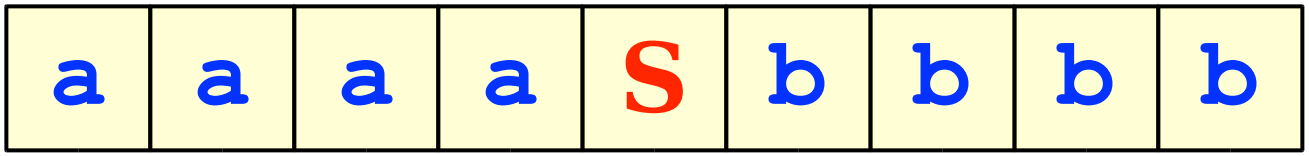

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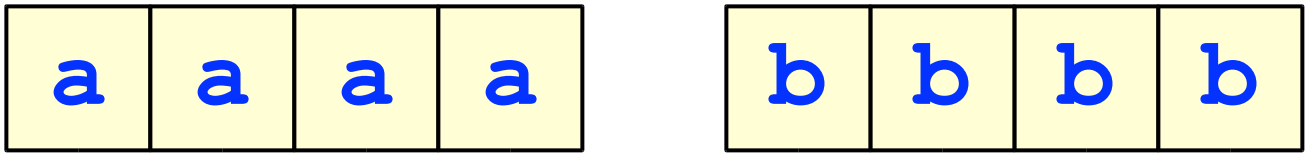

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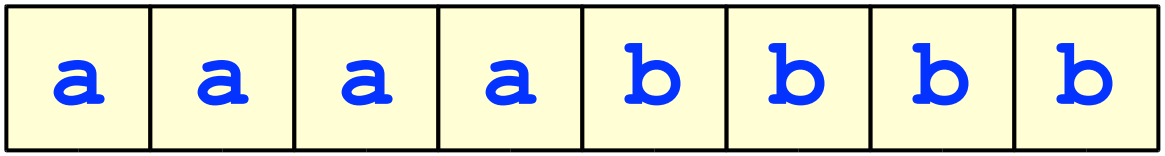

Time-Out for Announcements!

Second Midterm Logistics

- Our second midterm exam is next ***Tuesday, May 21st*** from ***7PM – 10PM***. Locations are divvied up by last (family) name:
  - `A – H`: Go to 200-002.
  - `I – Z`: Go to Bishop Auditorium.
- Topic coverage is primarily lectures 06 – 13 (functions through induction) and PS3 – PS5. Finite automata and onward won't be tested here.
  - Because the material is cumulative, topics from PS1 – PS2 and Lectures 00 – 05 are also fair game.
- The exam is closed-book and closed-computer. You can bring one double-sided 8.5" × 11" sheet of notes with you.
- Students with OAE accommodations: you should have heard from us with alternate exam locations. If you haven't, contact us ASAP.

Our Advice

- ***Stay fed and rested.*** You are not a brain in a jar. You are a rich, complex, beautiful human being. Please take care of yourself.
- ***Read all questions before diving into them.*** You don't have to go sequentially. Read over each problem so you know what to expect, then pick whichever one looks easiest and start there.
- ***Reflect on how far you've come.*** How many of these questions would you have been able to *understand* two months ago? That's the mark that you're learning something!

Back to CS103!

Designing CFGs

- Like designing DFAs, NFAs, and regular expressions, designing CFGs is a craft.
- When thinking about CFGs:
  - ***Think recursively:*** Build up bigger structures from smaller ones.
  - ***Have a construction plan:*** Know in what order you will build up the string.
  - ***Store information in nonterminals:*** Have each nonterminal correspond to some useful piece of information.
- Check our online “Guide to CFGs” for more information about CFG design.
- We’ll hit the highlights in the rest of this lecture.

Designing CFGs

- Let $\Sigma = \{$**a**, **b**$\}$ and let $L = \{w \in \Sigma^* \mid w$ is a palindrome $\}$
- We can design a CFG for L by thinking inductively:
  - Base case: ε, **a**, and **b** are palindromes.
  - If ω is a palindrome, then **a**ω**a** and **b**ω**b** are palindromes.
  - No other strings are palindromes.

S → **ε** | **a** | **b** | **a****S****a** | **b****S****b**

Designing CFGs

- Let $\Sigma = \{$**{**, **}**$\}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced braces $\}$
- Some sample strings in L:

{{{}}}

{{}}{}

{{}{}}{{}{}}

{{{{}}}{{}}}

ε

{}{}

Designing CFGs

- Let $\Sigma = \{$**{**, **}**$\}$ and let $L = \{w \in \Sigma^* \mid w$ is a string of balanced braces $\}$
- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace.

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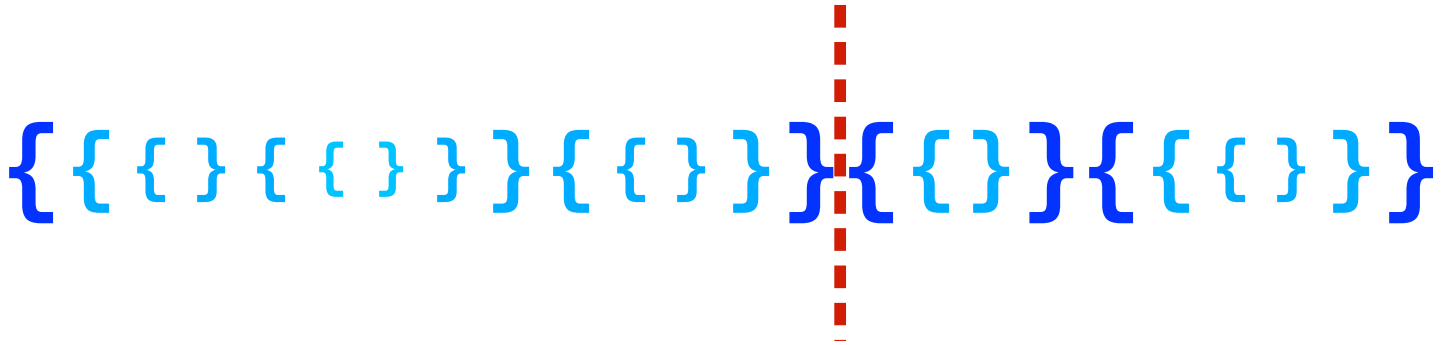

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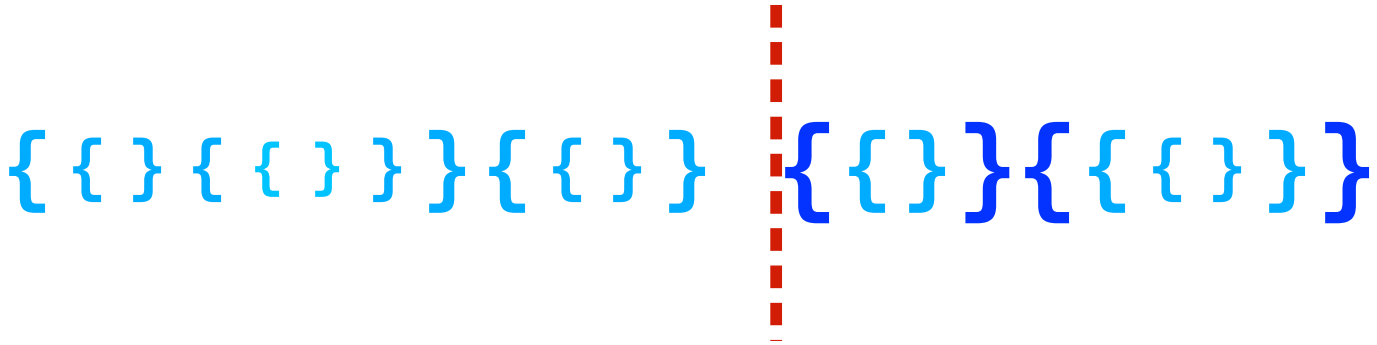

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- Let's think about this recursively.
  - Base case: the empty string is a string of balanced braces.
  - Recursive step: Look at the closing brace that matches the first open brace. Removing the first brace and the matching brace forms two new strings of balanced braces.

S → **{S}S** | **ε**

Designing CFGs

- Let $\Sigma = \{$ **a**, **b** $\}$ and let $L = \{ w \in \Sigma^* \mid w$ has the same number of **a**'s and **b**'s $\}$

Which of these CFGs have language L?

S → **a**S**b** | **b**S**a** | **ε**

S → **ab**S | **ba**S | **ε**

S → **ab**S**ba** | **ba**S**ab** | **ε**

S → S**ba**S | S**ab**S | **ε**

Answer at
https://cs103.stanford.edu/pollev

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S → S**ba**S | S**ab**S | **ε**

Designing CFGs: A Caveat

- When designing a CFG for a language, make sure that it
  - generates all the strings in the language and
  - never generates a string outside the language.
- The first of these can be tricky – make sure to test your grammars!
- You'll design your own CFG for this language on Problem Set 8.

CFG Caveats II

- Is the following grammar a CFG for the language $\{ \mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N} \}$?

$$\mathbf{S} \rightarrow \mathtt{a}\mathbf{S}\mathtt{b}$$

- What strings in $\{\mathtt{a}, \mathtt{b}\}^*$ can you derive?
  - Answer: ***None!***
- What is the language of the grammar?
  - Answer: ***Ø***
- When designing CFGs, make sure your recursion actually terminates!

Designing CFGs

- When designing CFGs, remember that each nonterminal can be expanded out independently of the others.
- Let $\Sigma = \{\mathtt{a}, \stackrel{?}{=}\}$ and let $L = \{\mathtt{a}^n \stackrel{?}{=} \mathtt{a}^n \mid n \in \mathbb{N}\}$.
- Is the following a CFG for L?

$$\mathtt{S} \to \mathtt{X} \stackrel{?}{=} \mathtt{X}$$

$$\mathtt{X} \to \mathtt{aX} \mid \varepsilon$$

$$\begin{aligned} & \mathtt{S} \\ \Rightarrow\ & \mathtt{X} \stackrel{?}{=} \mathtt{X} \\ \Rightarrow\ & \mathtt{aX} \stackrel{?}{=} \mathtt{X} \\ \Rightarrow\ & \mathtt{aaX} \stackrel{?}{=} \mathtt{X} \\ \Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathtt{X} \\ \Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathtt{aX} \\ \Rightarrow\ & \mathtt{aa} \stackrel{?}{=} \mathtt{a} \end{aligned}$$

Finding a Build Order

- Let $\Sigma = \{\mathtt{a}, \stackrel{?}{=}\}$ and let $L = \{\mathtt{a}^n \stackrel{?}{=} \mathtt{a}^n \mid n \in \mathbb{N}\}$.
- To build a CFG for L, we need to be more clever with how we construct the string.
  - If we build the strings of a's independently of one another, then we can't enforce that they have the same length.
  - ***Idea:*** Build both strings of a's at the same time.
- Here's one possible gramm
  that idea:

$$\mathbf{S} \to \stackrel{?}{=} \mid \mathtt{a}\mathbf{S}\mathtt{a}$$

$$\begin{aligned} & \mathbf{S} \\ \Rightarrow\ & \mathtt{a}\mathbf{S}\mathtt{a} \\ \Rightarrow\ & \mathtt{aa}\mathbf{S}\mathtt{aa} \\ \Rightarrow\ & \mathtt{aaa}\mathbf{S}\mathtt{aaa} \\ \Rightarrow\ & \mathtt{aaa}\stackrel{?}{=}\mathtt{aaa} \end{aligned}$$

Function Prototypes

- Let Σ = {`void`, `int`, `double`, `name`, `(`, `)`, `,`, `;`}.
- Let's write a CFG for C-style function prototypes!
- Examples:
  - `void name(int name, double name);`
  - `int name();`
  - `int name(double name);`
  - `int name(int, int name, int);`
  - `void name(void);`

Function Prototypes

- Here's one possible grammar:
  - **S** → **Ret** `name` `(` **Args** `);`
  - **Ret** → **Type** | `void`
  - **Type** → `int` | `double`
  - **Args** → `ε` | `void` | **ArgList**
  - **ArgList** → **OneArg** | **ArgList**, **OneArg**
  - **OneArg** → **Type** | **Type** `name`
- Fun question to think about: what changes would you need to make to support pointer types?

Summary of CFG Design Tips

- Look for recursive structures where they exist: they can help guide you toward a solution.
- Keep the build order in mind – often, you'll build two totally different parts of the string concurrently.
  - Usually, those parts are built in opposite directions: one's built left-to-right, the other right-to-left.
- Use different nonterminals to represent different structures.

Applications of Context-Free Grammars

CFGs for Programming Languages

BLOCK → **STMT**
| `{` **STMTS** `}`

STMTS → ε
| **STMT STMTS**

STMT → **EXPR** `;`
| `if (`**EXPR**`)` **BLOCK**
| `while (`**EXPR**`)` **BLOCK**
| `do` **BLOCK** `while (`**EXPR**`);`
| **BLOCK**
| …

EXPR → `identifier`
| `constant`
| **EXPR** `+` **EXPR**
| **EXPR** `-` **EXPR**
| **EXPR** `*` **EXPR**
| …

Grammars in Compilers

- One of the key steps in a compiler is figuring out what a program “means.”
- This is usually done by defining a grammar showing the high-level structure of a programming language.
- There are certain classes of grammars (LL(1) grammars, LR(1) grammars, LALR(1) grammars, etc.) for which it's easy to figure out how a particular string was derived.
- Tools like `yacc` or `bison` automatically generate parsers from these grammars.
- Curious to learn more? ***Take CS143!***

Natural Language Processing

- By building context-free grammars for actual languages and applying statistical inference, it's possible for a computer to recover the likely meaning of a sentence.
  - In fact, CFGs were first called ***phrase-structure grammars*** and were introduced by Noam Chomsky in his seminal work *Syntactic Structures*.
  - They were then adapted for use in the context of programming languages, where they were called ***Backus-Naur forms***.
- The **Stanford Parser** project is one place to look for an example of this.
- Want to learn more? Take CS124 or CS224N!

Next Time

- ***Turing Machines***
  - What does a computer with unbounded memory look like?
  - How would you program it?